JOURNAL OF APPROXIMATION THEORY 30, 129–138 (1980)

Hermite–Birkhoff Interpolation and Approximation of a Function and Its Derivatives*

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Communicated by E. W. Cheney

Received May 9, 1979

1.

In this paper we look at some fundamental results on Hermite and Hermite–Birkhoff interpolation by elements of Haar spaces and apply these to certain problems in the approximation of a function and its derivatives. Important interpolation results have been given by Ikebe [6] and Haussmann [4, 5]. We shall use the main theorem of [5] to generalize the main theorem of [6]. As an application we obtain information about the dimension of the set of best approximations of a function and its derivatives. An example is presented.

2.

A basic notion in approximation and interpolation theory is that of a Haar (or Tchebycheff) space. A subspace H of $C[a, b]$ of finite dimension n is called a Haar space if every element $h \in H$ has at most $n-1$ distinct zeros in $[a, b]$ or else vanishes identically. As is usual we shall say that $h \in C[a, b]$ has a zero of multiplicity k at $x_0 \in [a, b]$ if $h(x_0) = h'(x_0) = \cdots = h^{(k-1)}(x_0) = 0$ and either $h^{(k)}(x_0) \neq 0$ or does not exist. We say the multiplicity is ∞ if $h^{(i)}(x_0) = 0$ for all $i \geqslant 0$. A zero in (a, b) where h does not change sign will be called a nonnodal zero. Any other zero of h in $[a, b]$ will be called a nodal zero. This is the terminology of Karlin and Studden [9].

In Theorem 2 that follows, it will be convenient to use the terminology of Atkinson and Sharma [1]. Let $E = (e_{ij})$, $B = (b_{ij})$, $1 \leqslant i \leqslant r$, $0 \leqslant j \leqslant k$, where each entry of E is either 0 or 1 and the entries of B are given real numbers.

* Supported by NSERC of Canada Grant A-8755.

0021-9045/80/100129-10$02.00/0

Let $a_1 < a_2 < \cdots < a_r$ be points in $[a, b]$ that will be called nodes. The nodes and the matrices E and B determine a Hermite–Birkhoff interpolation problem for a subspace H of $C^{(k)}[a, b]$, the space of real-valued functions on $[a, b]$ with continuous kth derivatives. That is, we seek to find an $h \in H$, if it exists such that $h^{(j)}(a_i) = b_{ij}$ whenever $e_{ij} = 1$. The matrix E is called an incidence matrix. Note that not all the entries of B are relevant to the problem. For dim $H = n$, it is usually required that exactly n of the entries of E be 1, but we will not insist on this here. If for a given H the interpolation problem determined by a specific E and any set of r distinct nodes and any B, always has a solution, E is said to be poised. E is said to be conservative if whenever we have $e_{ij} = 0$, $e_{i,j+1} = e_{i,j+2} = \cdots = e_{i,j+q-1} = 1$ and $e_{i,j+q} = 0$ or is undefined, for some i, j and even q, then either $e_{uv} = 0$ for all $u > i$, $v \leqslant j$ or else $e_{uv} = 0$ for all $u > i$, $v \leqslant j$. An incidence matrix E satisfies the Pólya condition if for $q = 0, 1, \ldots, k$, the total number of 1's in columns 0 to q is at least $q + 1$. Finally, an interpolation problem will be called Hermite if the associated matrix $E = (e_{ij})$ has the property that $e_{ij} = 1$ implies $e_{iv} = 1$ for $0 \leqslant v \leqslant j$. A fuller discussion of the definitions and also some examples, can be found in [1, 6].

For $H \subset C^{(k)}[a, b]$ and i an integer, $0 \leqslant i \leqslant k$, define $H^{(i)}$ by $H^{(i)} \equiv \{h^{(i)}: h \in H\}$. H will always be a subspace in this paper.

3.

The following theorem is due to Haussmann [5]. The proof in the cited work is intertwined with some related results and a great deal of notation is introduced as a consequence. We supply a short, direct proof of the theorem. While it is arranged somewhat differently, the proof below uses the same ideas as in [5]. We begin with a preliminary lemma.

LEMMA. *For $k \geqslant 0$, let $f \in C^{(k)}[a, b]$ and suppose that $x_0 \in (a, b)$ is an isolated zero of $f, f', \ldots, f^{(k)}$. Also suppose that x_0 is a nonnodal zero of $f^{(k)}$. Let g_1, g_2 be functions in $C^{(k)}[a, b]$ such that*

(1) $g_1^{(i)}(x_0) = 0, \quad 0 \leqslant i \leqslant k - 1, \quad g_1^{(k)}(x_0) > 0,$

(2) $g_2^{(i)}(x_0) = 0, \quad 0 \leqslant i \leqslant k - 1, \quad g_2^{(k)}(x_0) < 0,$

where the homogeneous conditions are omitted if $k = 0$. Then given $0 < \varepsilon < \min\{b - x_0, x_0 - a\}$ there is an i, $1 \leqslant i \leqslant 2$ and a $\delta > 0$ such that $f + \delta^* g_i$ has two distinct zeros in the deleted neighbourhood $(x_0 - \varepsilon, x_0 + \varepsilon) \sim \{x_0\}$ for $0 < \delta^* \leqslant \delta$.

Proof. Assume first that k is even. Then f must have a nonnodal zero at

x_0. Then also assume $f(x) \geqslant 0$ in a neighbourhood of x_0. If g_2 satisfies (2) above and $\delta^* > 0$, then for $k \geqslant 2$, $f + \delta^* g_2$ has a nonnodal zero at x_0 and $(f + \delta^* g_2)(x) < 0$ in a deleted neighbourhood of x_0. If $k = 0$, $(f + \delta^* g_2)(x_0) < 0$. Using the intermediate value theorem and standard arguments, the required result follows. The case where $f(x) \leqslant 0$ in a neighbourhood of x_0 is treated similarly using g_1. Now assume that k is odd so that f has a nodal zero at x_0. Then also assume $f(x) > 0$ in a right neighbourhood of x_0 and $f(x) < 0$ in a left neighbourhood of x_0. Then if g_2 satisfies (2) above and $\delta^* > 0$, $f + \delta^* g_2$ has a nodal zero at x_0 and is such that $(f + \delta^* g_2)(x) < 0$ in a right neighbourhood of x_0 and $(f + \delta^* g_2)(x) > 0$ in a left neighbourhood of x_0. Again the required result follows from the intermediate value theorem. The last remaining case is handled similarly, using g_1. ∎

THEOREM 1 (Haussmann). *Let $H \subset C^{(k)}[a, b]$ and $H^{(1)}, H^{(2)}, \ldots, H^{(k)}$ be Haar spaces on $[a, b]$ with dimension $H = n$, n finite. Let $A_0 \supset A_1 \supset \cdots \supset A_k$ be subsets of $[a, b]$, where A_i has m_i elements $\{a_j^i\}$, $1 \leqslant j \leqslant m_i$ and $\sum_{i=0}^{k} m_i \leqslant n$. Further require that $\{a, b\} \cap A_1 = \varnothing$. Then given arbitrary real numbers $\{b_j^i\}$, $0 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m_i$, there is a function $h \in H$ such that $h^{(i)}(a_j^i) = b_j^i$, $0 \leqslant i \leqslant k$, $1 \leqslant j \leqslant m_i$. If $\sum_{i=0}^{k} m_i = n$, there is exactly one such function.*

Proof. The proof proceeds by induction. Clearly it suffices to show that for $\sum_{i=0}^{k} m_i = n$ and $b_j^i = 0$ for all i and j, the above interpolation problem has only the solution $h(x) \equiv 0$. The theorem is clearly true for $k = 0$ and any n, from the definition of Haar space. Assume that the theorem is true for $k = r$ and any n. Suppose for $k = r + 1$ that there is a function $h \in H$, $h(x) \not\equiv 0$ that satisfies a homogeneous interpolation problem of the type indicated above. First assume that every element in A_{r+1} is a nonnodal zero of $h^{(r)}$. By the lemma and the induction hypothesis it is possible to find $\bar{h} \in H$ such that

(i) $w(x) \equiv h(x) + \bar{h}(x) \not\equiv 0$,

(ii) $w^{(i)}(x) = 0$ for $0 \leqslant i \leqslant r$ and $x \in A_i \sim A_{r+1}$,

(iii) $w^{(i)}(x) = 0$ for $0 \leqslant i \leqslant r - 1$ and $x \in A_{r+1}$,

(iv) $\exists$ disjoint deleted neighbourhoods about each $x \in A_{r+1}$ where $w(x)$ has two distinct zeros not in A_0. We see that $w(x)$ has at least n zeros in $[a, b]$ where we have counted each interior zero a number of times equal to the minimum of its multiplicity and $r + 1$. Thus $w(x) \equiv 0$ by the induction hypothesis, a contradiction. Now assume that there is a zero x_0 of h in A_{r+1} that is a nodal zero of $h^{(r)}$. Observe that it must be a nonnodal zero of $h^{(r+1)}$. Using Rolle's theorem, h' has $n - 1$ zeros in (a, b) counting each zero as above. Let B represent the set of $m_0 - 1$ new zeros of h' introduced by the differentiation. Then by the lemma and the induction hypothesis, there is an $\bar{h}' \in H^{(1)}$ such that

(i) $w'(x) \equiv h'(x) + \bar{h}'(x) \not\equiv 0$ ($h' \not\equiv 0$ since the contrary implies $h \equiv 0$),

(ii) $w^{(i)}(x) = 0$, $1 \leqslant i \leqslant r + 1$, $x \in A_i - \{x_0\}$,

(iii) $w'(x) = 0$, $x \in B$.

(iv) $w^{(i)}(x_0) = 0$, $1 \leqslant i \leqslant r$,

(v) w' has two distinct zeros in a deleted neighbourhood of x_0, disjoint from $A_1 \cup B$. Thus w' has a total of n zeros, counting as above. Thus $w'(x) \equiv 0$ by the induction hypothesis, a contradiction. ∎

Remark. We observe that Theorem 1 and the lemma can be combined to show that a non-zero function h in a space H satisfying the conditions of Theorem 1 can have at most $n - 1$ zeros where interior zeros of multiplicity less than $k + 1$ are counted a number of times equal to their multiplicity and any interior zero x_0 of multiplicity $k + 1$ is counted $k + 1$ times if it is a nodal zero of $h^{(k)}$ and $k + 2$ times otherwise.

When differentiation reduces the dimension of $H^{(i)}$, $0 \leqslant i \leqslant k$, it is possible to make a stronger statement than that of Theorem 1. We shall combine Theorem 1 with the result of Ikebe [6] to obtain Theorem 2 that gives both of these results as special cases. The first part of the proof of Theorem 2 closely parallels the proof in [6], so this will only be sketched. We begin with two definitions.

Definition. Let E be an incidence matrix for some interpolation problem. A matrix formed from E by augmenting a last column consisting of zeros or ones, will be called an extended incidence matrix (for the interpolation problem).

Definition. Let C be an extended incidence matrix with p columns, $p \geqslant 3$. A matrix C^* will be called derivable from C if it can b obtained from C in the following way: whenever a row i of C begins with a 1 and for minimal $j > i$, row j begins with a 1, either add to C anywhere between row i and row j, a row that has a 1 in the second position and zeros elsewhere, or alter a row of C strictly between row i and row j by changing the second zero that appears in this row to a 1. Call the new matrix so obtained $\bar{C}$. Form C^* by deleting the first column of $\bar{C}$.

Theorem 2. *Let $H \subset C^{(k)}[a, b]$ and $H^{(1)}, H^{(2)}, \ldots, H^{(k)}$ be Haar spaces on $[a, b]$. Let m be an integer $0 \leqslant m \leqslant k$ and require that for an integer $n \geqslant m$,* $\dim H^{(i)} = n - i$, *where $i \leqslant m$. If $m < k$ require further that for $m \leqslant i \leqslant k$ the functions in $H^{(i)}$ can be extended so as to form a Haar space with dimension*

$\dim H^{(i)}$ on some open interval containing $[a,b]$. Let $E=(e_{ij})$ be an incidence matrix for a Hermite–Birkhoff interpolation problem using H, $H^{(1)},\dots,H^{(k)}$. Require that the following conditions hold:

(i) *E satisfies the Pólya condition,*

(ii) *E is conservative,*

(iii) *Whenever $j\geqslant m+1$ and $e_{ij}=1$ then $e_{i,j-1}=1$,*

(iv) *E has exactly n entries of "1."*

Then E is posed.

Proof. Suppose there is an $h\in H$, $h(x)\not\equiv 0$ that satisfies a homogeneous interpolation problem with incidence matrix E. Form the extended incidence matrix B by adjoining a column of zeros to the right of E. By Rolle's theorem h' has a nodal zero between each zero of h. Then h' must satisfy an interpolation problem with an extended incidence matrix with $n-1$ "1" entries, that is derivable from B. Specifically, calling such a matrix B_1, we mean that h' satisfies a homogeneous interpolation problem with incidence matrix $\bar{B}_1$, defined to be B_1 with the last column omitted. Also, whenever there is a one in the last two positions of a row in B_1 this is interpreted to mean that $h^{(k)}$ has a nonnodal zero at the point corresponding to this row. It is to make this interpretation that the extended incidence matrix was introduced. We may verify that $\bar{B}_1$ satisfies (i)–(iii) and that the extended incidence matrix B_1 has $n-1$ entries of "1." Proceeding similarly, we obtain a sequence of matrices $B_2, B_3,\dots, B_m$. Now B_m is in Hermite form and has $n-m$ elements. $H^{(m)}$ has dimension $n-m$. By the remark following Theorem 1 we see that $h^{(m)}\equiv 0$. Since E, $\bar{B}_1, \bar{B}_2,\dots,\bar{B}_{m-1}$ satisfy the Pólya condition we may deduce that $h(x)\equiv 0$, a contradiction and the theorem follows. ∎

4.

Consider now the problem of finding a best approximation h from a subspace $H\subset C^{(k)}[a,b]$ to a function $f\in C^{(k)}[a,b]$ using the norm $|||\cdot|||$ defined by

$$|||f|||=\max\{\|f\|, \|f'\|,\dots, \|f^{(k)}\|\},$$

where $\|\cdot\|$ represents the sup norm on $[a,b]$.

This problem has been treated, especially when H is a space of polynomials, by many authors: [2, 3, 7, 8, 10–12, 14, 16].

DEFINITION. Let $f\in C^{(k)}[a,b]$. If $x_0\in[a,b]$ is such that for $h\in C^{(k)}[a,b]$ and some i, $0\leqslant i\leqslant k$, $|h^{(i)}(x_0)-f^{(i)}(x_0)|=|||h-f|||$, call x_0 an i-extreme point of $h-f$.

We may observe that existence of best approximations is readily demonstrated when H is finite dimensional.

We can now obtain the following theorem. Some of the ideas used are similar to those of Johnson and Schuurman.

THEOREM 3. *Let* $f \in C^{(k+1)}[a,b]$, $k \geqslant 2$ *and let* $H \subset C^{(k+1)}[a,b]$ *be a Haar space of dimension n such that* $\dim H^{(i)} = n - i$ *for* $i \leqslant k-1$. *Require also that* $H^{(1)}, H^{(2)}, \ldots, H^{(k+1)}$ *be Haar on* $[a,b]$. *If* $\dim H^{(k)} = n - k + 1$, *require also that* $H^{(k-1)}$, $H^{(k)}$ *and* $H^{(k+1)}$ *be restrictions to* $[a,b]$ *of Haar spaces of similar dimension on some open interval containing* $[a,b]$. *Then the dimension of the (convex) set of best approximations to f using norm* $|||\cdot|||$ *is at most* $k+1$. *If* $\dim H^{(k)} = n-k$, *and if* $H^{(k)}$ *and* $H^{(k+1)}$ *are restrictions to* $[a,b]$ *of Haar spaces of similar dimension on some open interval containing* $[a,b]$, *then we can further deduce that if p and q are best approximations to f, then* $p^{(k)} = q^{(k)}$.

Proof. Assume that p and w, $p \neq w$, are two functions of best approximation in the relative interior of the set of best approximations. Then every i-extreme point of $w-f$ is an i-extreme point of $p-f$, with error of similar sign. If x_0 is an interior i-extreme point of $w-f$, then since necessarily $w^{(i+1)}(x_0) = f^{(i+1)}(x_0)$, x_0 is not an $(i+1)$-extreme point of $w-f$. Note, however, that for such x_0, $p^{(i)}(x_0) - w_i(x_0) = p^{(i+1)}(x_0) - w^{(i+1)}(x_0) = 0$. Denote the number of boundary i-extreme points of $w-f$ by α_i and the number of interior i-extreme points of $w-f$ by β_i. Let $Q \equiv \{M \leqslant k-2: \sum_{i=j}^{M} (\alpha_i + 2\beta_i) \leqslant M - j$ for all j such that $0 \leqslant j \leqslant M\}$. Let $d \equiv \operatorname{card} Q$. Let $R = \{x \in \{a,b\} : x$ is not a $(k-1)$-extreme point of $w-f\}$. Let $g \equiv \operatorname{card} R$. Then $m \equiv \sum_{i=0}^{k} \alpha_i + 2\sum_{i=0}^{k} \beta_i + d + g \geqslant n+1$. Indeed since w is best, there can be no $z \in H$ such that for each i, $0 \leqslant i \leqslant k$, $\operatorname{sgn} z^{(i)}(x) = \operatorname{sgn}(f^{(i)}(x) - w^{(i)}(x)) \neq 0$ whenever x is an i-extreme point of $w-f$. Supposing $m \leqslant n$ form the following interpolation problem:

(1) $z^{(i)}(x) = \operatorname{sgn}(f^{(i)}(x) - w^{(i)}(x))$ if x is an i-extreme point of $w-f$.

(2) $z^{(i-1)}(x) = 0$ if x is an interior i-extreme point of $w-f$ and either $i < k$ or $i = k$ and x is a $(k-2)$ extreme point of $w-f$,

(3) $z^{(k-1)}(x) = 0$ if x is an interior k-extreme point of $w-f$ but not a $(k-2)$-extreme point of $w-f$,

(4) $z^{(i)}(b) = 0$ if $i \in Q$,

(5) $z^{(k-1)}(x) = 0$ if $x \in R$.

Clearly, the number of conditions to be satisfied is $\sum_{i=0}^{k} \alpha_i + 2\sum_{i=0}^{k} \beta_i + d + g = m \leqslant n$. By Theorem 2, the above interpolation problem has a solution, a contradiction. Note that (2) and (3) above serve to satisfy condition (ii) of Theorem 2. Conditions (3) and (5) serve to satisfy (iii) and

(4) serves to satisfy (i) of that theorem. Note further that if $\dim H^{(k)} = n - k$, consideration of the interpolation problem with (5) omitted and similar arguments allows us to deduce that $\sum_{i=0}^{k} \alpha_i + 2\sum_{i=0}^{k} \beta_i + d \geqslant n + 1$. Now consider $r = p - w$. If x_0 is an interior i-extreme point of $w - f$, $0 \leqslant i \leqslant k$, then $r^{(i)}$ has a zero of multiplicity at least 2 at x_0. If x_0 is a boundary i-extreme point of $w - f$, $0 \leqslant i \leqslant k$, $r^{(i)}$ has a zero at x_0. By differentiating r k times and mimicking the proof of Theorem 2, we may deduce that if $\dim H^{(k)} = n - k$, $r^{(k)}$ has at least $\sum_{i=0}^{k} \alpha_i + 2\sum_{i=0}^{k} \beta_i - k + d \geqslant n - k + 1$ zeros, counting nonnodal zeros twice and nodal zeros once. The number d is introduced since there are at least d differentiations where the number of zeros of the derived function is not reduced. We see that $r^{(k)} \equiv 0$ as required. Suppose now that $\dim H^{(k)} = n - k + 1$. Then if we add the g conditions on $r^{(k-1)}$, namely $r^{(k-1)}(x) = 0$ if $x \in R$, we find that $r^{(k)}$ has $\sum_{i=0}^{k} \alpha_i + 2\sum_{i=0}^{k} \beta_i - k + d + g \geqslant n - k + 1$ zeros counting as before. Thus $r^{(k)}$ and hence $r^{(k-1)}$ are identically zero. The set $T \equiv \{r^{(k-1)}: r = w - p;\ p$ a best approximation to $f\}$ is convex and contains 0 as a relative interior point. We see that this set intersects the space $V \equiv \{h \in H^{(k-1)}: h(x_0) = 0$ for $x_0 \in R\}$ only at 0. Since V considered as a subspace of $H^{(k-1)}$ has co-dimension $g \leqslant 2$, it follows that T has dimension $\leqslant 2$ and that the space of best approximations has dimension $\leqslant k + 1$. ∎

The theorem does not include the case $k = 1$. In fact a stronger result can be given in this situation.

THEOREM 4. *Let $f \in C^{(2)}[a, b]$ and let $H \subset C^{(2)}[a, b]$ be a Haar space of dimension n. If $H^{(1)}$ is a Haar space of dimension $n - 1$ on $[a, b]$ and if p and q are best approximations to f from H in* norm $|||\cdot|||$, $k = 1$, *then $p(x) \equiv q(x) + C$ for some constant C. If $H^{(1)}$ is the restriction of an n-dimensional Haar space on an open interval containing $[a, b]$, then the set of best approximations to f from H in* norm $|||\cdot|||$ *has dimension at most one.*

Proof. An analysis of Moursund's result [14] on the special case of polynomial approximation shows that his proof depends only on the fact that $\dim H^{(1)} = n - 1$ and the ability to do Hermite interpolation with polynomials. As we have seen, the latter property is a special case of Ikebe's theorem. (Indeed for $k = 1$ this result is also given in Schuurman [16], although in the literature it is consistently attributed to Matthews [13] whose paper appeared later.) Thus Moursund's argument can be used to establish the first part of the theorem. Now assume that $H^{(1)}$ is the restriction of an n-dimensional Haar space. Suppose that p and w are best approximations, w in the relative interior of the set of best approximations. Let $E_0 = \{x \in [a, b]: x$ is an 0-extreme point of $w - f\}$ and let $E_1 = \{x \in [a, b]: x$ is a 1-extreme point of $w - f\}$. Now $\mathrm{card}(E_0 \cup E_1) + \mathrm{card}(E_1) \geqslant n + 1$ by Haussmann's theorem and arguments previously employed in Theorem 3. Let $r = p - w$.

Then r has at least a double zero at all $x \in E_0$ except possibly for $x \in \{a, b\}$. Similarly r' has a double zero at all $x \in E_1$ except possibly for $x \in \{a, b\}$. By considering the various possibilities for a and b we may deduce that there are sets S and T, $S \subset E_0 \cup E_1$, $T \subset S$ where r' is zero on S and r'' is zero on T and $\operatorname{card} S + \operatorname{card} T \geqslant \operatorname{card}(E_0 \cup E_1) + \operatorname{card} E_1 - 2 \geqslant n - 1$. By an argument similar to that used at the end of the proof of Theorem 3, we may deduce that the space of best approximations is at most one-dimensional.

5.

We present an example where the space of best approximations in norm $|||\cdot|||$, $k = 1$, is one-dimensional, but where pairs of best approximations do not differ by a constant.

EXAMPLE. Let $H = \operatorname{span}\{e^x, e^{2x}\}$ with domain taken to be $[1, 1 + \ln 2]$. Clearly $H = H^{(1)}$. H is known to be a Haar space [15]. Denote $1 + \ln 2$ by b. Let g be a differentiable function defined on $[1, 1.01]$ such that $g(1) = g'(1) = 0$, $0 \leqslant g'(x) < 1$ on its domain and such that $g(1.01) > 0.0099$. Such a function clearly exists. Define f on $[1, b]$ by

$$\begin{aligned} f(x) &= e^x + 1 - (x - b)^2, && x \in [b - 0.01, b], \\ &= e^x + 0.9999, && x \in [1.01, b - 0.01], \\ &= e^x - x + g(x) + 2.0099 - g(1.01), && x \in [1, 1.01]. \end{aligned}$$

One may verify that

(1) $f(b) = e^b + 1$,

(2) $|f(x) - e^x| < 1$ if $x \in [1, b)$,

(3) $f'(1) = e - 1$,

(4) $|f'(x) - e^x| < 1$ if $x \in [1, b) - \{1, 1.01, b - 0.01\}$.

As defined, f fails to be differentiable at 1.01 and $b - 0.01$, but we may find a function $\bar{f}$ such that

(5) $\bar{f} \in C^{(2)}[1, b]$,

(6) $\bar{f}(x) = f(x)$ if $x \in [1, 1.01] \cup [b - 0.01, b]$,

(7) $|\bar{f}(x) - e^x| < 1$ if $x \in [1.01, b - 0.01]$,

(8) $|\bar{f}'(x) - e^x| < 1$ if $x \in [1.01, b - 0.01]$.

Now $y_1(x) = e^x$ is a best function-derivative approximation to $\bar{f}$ from H. Indeed if y_2 were better, then $y(x) = y_1(x) - y_2(x) < 0$ at $x = b$ and

$y'(x) = y_1'(x) - y_2'(x) > 0$ at $x = 1$. Suppose $y(x) = c_1 e^x + c_2 e^{2x}$. Then c_1 and c_2 must be chosen so that

$$c_1 e^b + c_2 e^{2b} < 0$$

and

$$c_1 e^1 + 2c_2 e^2 > 0,$$

i.e.,

$$c_1 \cdot 2e + c_2 \cdot 4e^2 > 0$$

and

$$c_1 \cdot e + c_2 \cdot 2e^2 < 0$$

which is clearly impossible.

Consider the function $\hat{y}(x) = e^{2x} - 2e^{x+1}$, $\hat{y} \in H$. Note that $\hat{y}(x)$ is 0 at $x = b$ and $\hat{y}'(x)$ is 0 at $x = 1$. Also, $\hat{y}'(b) = 4e^2 > 0$ and $\hat{y}''(1) = 2e^2 > 0$. Therefore for $c > 0$, $|y_1(x) - c\hat{y}(x) - \bar{f}(x)| \leqslant 1$ in some left neighbourhood of $x = b$ and $|y_1'(x) - c\hat{y}'(x) - \bar{f}'(x)| \leqslant 1$ in some right neighbourhood of $x = 1$. It follows that for sufficiently small $c > 0$,

$$y_c(x) = e^x - c(e^{2x} - 2e^{x+1})$$

is also a best approximation to f, and indeed all best approximations must be of this form.

Remark. The situation of seeking a best approximation in norm $|||\cdot|||$, $k = 1$, from a space of the form $H = \text{sp}\{e^{\lambda_1 x}, e^{\lambda_2 x}\}$, $\lambda_1 \neq \lambda_2$; $\lambda_1, \lambda_2 \neq 0$ has an unusual feature. It was crucial that in the example of the previous section the intervals have length exactly ln 2. Indeed, refer to the proof of Theorem 4. A short argument shows that for the special case of H given above, except in the instance of an interval of length $(\ln|\lambda_2| - \ln|\lambda_1|)/(\lambda_2 - \lambda_1)$, we may improve the theorem to show that the set of best approximations has dimension 0, i.e., there is exactly one best approximation. This improves on the results in [11] and explains the unusual observations in that discussion.

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